Degree Sequences and Majorization

Srinivasa R. Arikati* and Uri N. Peled†

Department of Mathematics, Statistics, and Computer Science
851 S. Morgan (M/C 249)
University of Illinois at Chicago
Chicago, IL 60607-7045

Submitted by Richard A. Brualdi

Dedicated to Ingram Olkin

ABSTRACT

A classical result concerning majorization is: given two nonnegative integer sequences a and b such that a majorizes b, a rearrangement of b can be obtained from a by a sequence of unit transformations. A recent result says that a degree sequence is a threshold sequence (degree sequence of a threshold graph) if and only if it is not strictly majorized by any degree sequence. Motivated by this, we define the majorization gap of a degree sequence to be the minimum number of successive reverse unit transformations required to transform it into a threshold sequence. We derive a formula for the majorization gap by establishing a lower bound for it and exhibiting reverse unit transformations achieving the bound. We also discuss the relationship between the majorization gap and the threshold gap (introduced elsewhere), and show that they are equal. The degree sequences having the maximum majorization gap for a fixed number of edges or vertices are characterized.

1. INTRODUCTION

Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be integer sequences of length n. Denote the ith largest component of a (b) by $a_{[i]}$ $(b_{[i]})$. We say that a **majorizes** b, denoted by $a \succcurlyeq b$, if

$$\sum_{i=1}^{k} a_{[i]} \geq \sum_{i=1}^{k} b_{[i]}, \qquad k = 1, \ldots, n,$$

*E-mail: arikati@mpi-sb.mpg.de.

†E-mail: peled@math.uic.edu.

655 Avenue of the Americas, New York, NY 10010 0024-3795/94/$7.00

with equality for $k = n$. The majorization is **strict**, denoted by $a \succ b$, if at least one of the inequalities is strict. If $a_i \geq a_j + 2$ for some i and j, we say that $c = a - u_i + u_j$ is obtained from a by a **unit transformation** from i to j, where u_i is the ith unit vector. We also say that a is obtained from c by a **reverse unit transformation** from j to i. Clearly $a \succcurlyeq c$. A classical result [11, 14] is that if $a \succcurlyeq b$, then some rearrangement of the components of b can be obtained from a by a finite number of successive unit transformations.

A **threshold graph** is a simple graph G with the property that there is a hyperplane separating the characteristic vectors of the independent sets of vertices of G from the characteristic vectors of the nonindependent sets. Several characterizations of the threshold graphs are known; see, for example, [2, 6].

A **threshold sequence** is the degree sequence of a threshold graph. One of the characterizations of threshold sequences is that the degree sequence has a unique realization as a labeled graph [8]. Another characterization is that they are the extreme points of the convex hull of all degree sequences of length n and their rearrangements [13, 15]. A third characterization is that they are not strictly majorized by any degree sequence [15]. Thus every degree sequence can be transformed into a threshold sequence by successive reverse unit transformations.

Motivated by the above results, we propose a measure of the nonthresholdness of a degree sequence. The **majorization gap** of a degree sequence d, denoted by $R(d)$, is defined as the minimum number of successive reverse unit transformations required to transform d into a threshold sequence. By definition $R(d) = 0$ if and only if d is a threshold sequence. We prove a formula for $R(d)$ in Section 3. It is fairly simple to show that the formula gives a lower bound for $R(d)$, and we exhibit successive reverse unit transformations that achieve this bound.

Hammer et al. [8] introduced the threshold gap of a degree sequence d and showed that it is half of the minimum L_1-distance between d and any threshold sequence of the same length. They also characterized the threshold sequences that achieve the threshold gap. We prove in Section 4 that the majorization gap and the threshold gap are equal. Furthermore, any threshold sequence that achieves the majorization gap also achieves the threshold gap.

The problem of characterizing the degree sequences with maximum majorization gap is discussed in Section 5. For a fixed number of edges, $R(d)$ is maximized precisely when d is the degree sequence of a matching plus isolated vertices. For a fixed number n of vertices, we exhibit all the degree sequences that maximize $R(d)$, and they turn out to be almost $n/2$-regular.

Section 6 discusses the difference gap—the bipartite analog of the majorization gap—and gives a formula for it. Section 7 concludes with some other related problems.

Throughout this paper, we represent degree sequences by corrected Ferrers diagrams. This and other related notions are the topic of Section 2.

2. PRELIMINARIES

A sequence of integers $d = (d_1, \ldots, d_n)$ such that $n - 1 \geq d_1 \geq \cdots \geq d_n \geq 0$ is called a **proper sequence**, after [8]. The **corrected conjugate sequence** [1] of a proper sequence d is the sequence $d' = (d'_1, \ldots, d'_n)$ given by

$$d'_k = |\{i : i < k \text{ and } d_i \geq k-1\}| + |\{i : i > k \text{ and } d_i \geq k\}|$$

(d' need not be proper, but it is "almost" proper; see Lemma 12). The corrected conjugate sequence may be represented by a **corrected Ferrers diagram** as follows. The corrected Ferrers diagram of a proper sequence d is a $(0, 1, \star)$-valued $n \times n$ matrix $C = C(d)$ whose main diagonal contains $\star$'s and the other $n - 1$ entries of whose ith row contain exactly d_i 1's, left-justified. Then d'_k is the number of 1's in the kth column of $C(d)$.

EXAMPLE 1. Let $d = (2,2,2,2,2)$. See Figure 1. Then $d' = (4,4,2,0,0)$.

A reverse unit transformation on a proper sequence d is represented on $C(d)$ by a transfer of the last 1 of a row to the end of a higher row. We make sure that the former row is the lowest among all rows with the same number of 1's, and similarly the latter row is highest. This prevents any row from becoming shorter than the next row, and thus the resulting sequence is proper.

For any sequence $d = (d_1, \ldots, d_n)$ let $S_k(d)$ denote the kth partial sum of d, i.e., $S_k(d) = \sum_{i=1}^{k} d_i$.

$$\begin{array}{ccccc}
\star & 1 & 1 & 0 & 0 \\
1 & \star & 1 & 0 & 0 \\
1 & 1 & \star & 0 & 0 \\
1 & 1 & 0 & \star & 0 \\
1 & 1 & 0 & 0 & \star
\end{array}$$

FIG. 1. The corrected Ferrers diagram $C(d)$ of the proper sequence $d = (2, 2, 2, 2, 2)$. The corrected conjugate sequence is $d' = (4, 4, 2, 0, 0)$.

A sequence $d = (d_1, \ldots, d_n)$ is called a **degree sequence** if there exists a (simple) graph $G = (V, E)$ on the vertex set $V = \{1, 2, \ldots, n\}$ such that $\deg(i) = d_i$ for all i. G is said to be a **realization** of d.

For a proper sequence d of length n, the following are equivalent (see [12, 7, 3, 1] for details):

(1) d is a degree sequence;
(2) $S_n(d)$ is even and d satisfies the *Erdős-Gallai inequalities*

$$S_k(d) \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k), \qquad k = 1, 2, \ldots, n;$$

(3) $S_n(d)$ is even and $S_k(d) \leq S_k(d')$ for $k = 1, 2, \ldots, n$;
(4) $(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ is also a degree sequence.

We shall be using the characterization (3) extensively. Note that for a proper sequence d, $S_n(d) = S_n(d')$.

A particularly useful characterization of threshold sequences is the following.

THEOREM 2. [8] *A proper degree sequence d is threshold if and only if $d = d'$, i.e., $C(d)$ is symmetric.*

For positive integers n, we define

$$D_n = \{d = (d_1, \ldots, d_n) : d \text{ is proper and } S_k(d) \leq S_k(d') \text{ for } k = 1, \ldots, n\}.$$

Thus d is a proper degree sequence of length n if and only if $d \in D_n$ and $S_n(d)$ is even.

The motivation for the current work is the following two theorems, already mentioned in Section 1.

THEOREM 3. [11, 14] *If a and b are integer sequences such that $a \succcurlyeq b$, then some rearrangement of b can be obtained from a by a finite sequence (possibly empty) of unit transformations.*

THEOREM 4. [15] *A degree sequence d is a threshold sequence if and only if d is not strictly majorized by any degree sequence.*

Theorem 4 states that

(1) every nonthreshold degree sequence is strictly majorized by some threshold sequence;
(2) threshold sequences are majorized only by their own rearrangements.

Further, Theorems 3 and 4 imply that if a degree sequence d is not threshold, then there exists a threshold sequence e such that (a rearrangement of) e can be obtained from d by successive reverse unit transformations. We are thus led to the definition of the majorization gap $R(d)$ as in the Introduction. In order to find an explicit expression for it, we use the following notation.

For a proper sequence $d = (d_1, \ldots, d_n)$, define

$$\delta_i(d) = (d_i' - d_i)^+, \qquad i = 1, \ldots, n,$$

where $x^+ = \max(x, 0)$, and

$$\delta(d) = \sum_{i=1}^{n} \delta_i(d).$$

Our first main result is a formula for the majorization gap, to be proved in the next section:

THEOREM 5. *For any proper degree sequence d,*

$$R(d) = \frac{\delta(d)}{2}.$$

EXAMPLE 6. Let $d = (2{,}2{,}2{,}2{,}2)$. Then $d' = (4, 4, 2, 0, 0)$ and $\delta(d) = 4$. The theorem asserts that $R(d) = 2$. A reverse unit transformation from 5 to 1 transforms d into $f = (3, 2, 2, 2, 1)$, and a reverse unit transformation from 4 to 1 transforms f into $e = (4, 2, 2, 1, 1)$. Since $e = e'$, e is a threshold sequence.

In the rest of this section we discuss some preliminary results.

For any two sequences $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$, define

$$\delta_i(a, b) = (a_i - b_i)^+, \qquad i = 1, \ldots, n,$$

and

$$\delta(a, b) = \sum_{i=1}^{n} \delta_i(a, b).$$

If a and b are integer sequences, $a \succcurlyeq b$, $a_1 \geq \cdots \geq a_n$, and $b_1 \geq \cdots \geq b_n$, define $U(a, b)$ to be the minimum number of successive unit transformations required to transform a into b. Observe that under these conditions the following are equivalent: (i) $a = b$; (ii) $\delta(a, b) = 0$; (iii) $U(a, b) = 0$.

LEMMA 7. *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be integer sequences such that $a \succcurlyeq b$, $a_1 \geq \cdots \geq a_n$, and $b_1 \geq \cdots \geq b_n$. Then $U(a, b) = \delta(a, b)$.*

Proof. It is easy to see that if $a \neq b$ and c is obtained from a by a unit transformation, then $\delta(a,b)+1 \geq \delta(c,b) \geq \delta(a,b)-1$. Thus $U(a,b) \geq \delta(a,b)$. Also, as given on p. 135 of [14], if $a \neq b$, we can always perform a unit transformation on a to obtain a c such that $c \not\succeq b$ and $\delta(c,b) = \delta(a,b) - 1$. Thus $U(a,b) \leq \delta(a,b)$. ■

It is easy to check that for integer x,

$$(x+1)^+ - x^+ = \begin{cases} 1, & x \geq 0, \\ 0, & \text{otherwise}, \end{cases} \qquad (x-1)^+ - x^+ = \begin{cases} -1, & x \geq 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$(x+2)^+ - x^+ = \begin{cases} 2, & x \geq 0, \\ 1, & x = -1, \\ 0, & \text{otherwise}, \end{cases} \qquad (x-2)^+ - x^+ = \begin{cases} -2, & x \geq 2, \\ -1, & x = 1, \\ 0, & \text{otherwise}. \end{cases} \tag{1}$$

We use these facts to prove the following lemma.

LEMMA 8. *Let $d = (d_1, \ldots, d_n)$ be a proper sequence, and assume that e is obtained from d by a reverse unit transformation. Then*

$$\delta(d) - 2 \leq \delta(e) \leq \delta(d) + 2.$$

Proof. Represent d by its corrected Ferrers diagram and assume that $e = d - u_\rho + u_\pi$. Let

$$\sigma = \begin{cases} d_\pi + 1 & \text{if } d_\pi < \pi - 1, \\ d_\pi + 2 & \text{if } d_\pi \geq \pi - 1, \end{cases}$$

$$\tau = \begin{cases} d_\rho & \text{if } d_\rho < \rho, \\ d_\rho + 1 & \text{if } d_\rho \geq \rho. \end{cases}$$

Thus the transfer is from row ρ, column τ of $C(d)$ to row π, column σ, and consequently $\pi \neq \sigma$ and $\rho \neq \tau$. See Figure 2.

We have four cases.

Case 1: $\pi \neq \tau$ and $\rho \neq \sigma$. Since $e = d - u_\rho + u_\pi$, for $1 \leq i \leq n$ we have $e_i = d_i$ except for $e_\pi = d_\pi + 1$, $e_\rho = d_\rho - 1$, and $e'_i = d'_i$ except for $e'_\sigma = d'_\sigma + 1$, $e'_\tau = d'_\tau - 1$. Then $\delta_\pi(e) = (e'_\pi - e_\pi)^+ = (d'_\pi - d_\pi - 1)^+$, and $\delta_\sigma(e) = (e'_\sigma - e_\sigma)^+ = (d'_\sigma - d_\sigma + 1)^+$. Hence

$$\delta_\pi(e) - \delta_\pi(d) = \begin{cases} -1 & \text{if } d'_\pi - d_\pi \geq 1, \\ 0 & \text{otherwise}, \end{cases}$$

$$\delta_\sigma(e) - \delta_\sigma(d) = \begin{cases} 1 & \text{if } d'_\sigma - d_\sigma \geq 0, \\ 0 & \text{otherwise}. \end{cases}$$

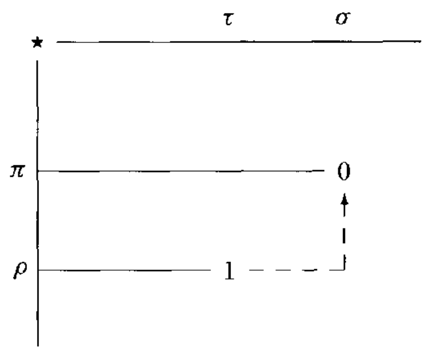

FIG. 2. Illustrating a transfer from ρ to π. Solid lines represent 1's and possibly $\star$'s.

Similarly,

$$\delta_\rho(e) - \delta_\rho(d) = \begin{cases} 1 & \text{if } d'_\rho - d_\rho \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta_\tau(e) - \delta_\tau(d) = \begin{cases} -1 & \text{if } d'_\tau - d_\tau \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Case 2: $\pi \neq \tau$ and $\rho = \sigma$. Here, $\delta_\pi(e) - \delta_\pi(d)$ and $\delta_\tau(e) - \delta_\tau(d)$ are as in case 1. Also $\delta_\rho(e) = (e'_\rho - e_\rho)^+ = (d'_\rho - d_\rho + 2)^+$. Hence

$$\delta_\rho(e) - \delta_\rho(d) = \begin{cases} 2 & \text{if } d'_\rho - d_\rho \geq 0, \\ 1 & \text{if } d'_\rho - d_\rho = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Case 3: $\pi = \tau$ and $\rho \neq \sigma$. Here $\delta_\rho(e) - \delta_\rho(d)$ and $\delta_\sigma(e) - \delta_\sigma(d)$ are as in case 1. Since $\delta_\pi(e) = (e'_\pi - e_\pi)^+ = (d'_\pi - d_\pi - 2)^+$, we obtain

$$\delta_\pi(e) - \delta_\pi(d) = \begin{cases} -2 & \text{if } d'_\pi - d_\pi \geq 2, \\ -1 & \text{if } d'_\pi - d_\pi = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Case 4: $\pi = \tau$ and $\rho = \sigma$. We now have $\delta_\pi(e) - \delta_\pi(d)$ as in case 3, and $\delta_\rho(e) - \delta_\rho(d)$ as in case 2.

To complete the proof, note that $\delta_i(e) = \delta_i(d)$ except for $i \in \{\pi, \rho, \sigma, \tau\}$. ∎

For future reference, we make the following observations from the proof of Lemma 8. Necessary and sufficient conditions for $\delta(e) = \delta(d) - 2$ in cases 1,2,3 are:

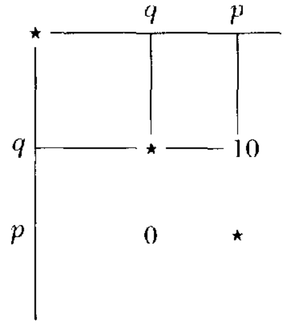

FIG. 3. Illustrating part 1 of Lemma 10.

(1) If $\pi \neq \tau$, $\rho \neq \sigma$, then (a) $d'_\pi - d_\pi \geq 1$, (b) $d'_\rho - d_\rho < 0$, (c) $d'_\sigma - d_\sigma < 0$, (d) $d'_\tau - d_\tau \geq 1$.
(2) If $\pi \neq \tau$, $\rho = \sigma$, then (a) $d'_\pi - d_\pi \geq 1$, (b) $d'_\rho - d_\rho \leq -2$, (c) $d'_\tau - d_\tau \geq 1$.
(3) If $\pi = \tau$, $\rho \neq \sigma$, then (a) $d'_\pi - d_\pi \geq 2$, (b) $d'_\rho - d_\rho < 0$, (c) $d'_\sigma - d_\sigma < 0$.

LEMMA 9. *Let d be a proper sequence. For $1 \leq i, j \leq n$, if $d_i < j - 1$, then $d'_j < i$.*

Proof. The proof is simple. ■

LEMMA 10. *Let $d \in D_n$, and let p be the largest index i such that $d_i < d'_i$. Then $p < n$. Further:*

(1) *Assume $d'_p \leq p - 1$, and let $q = d'_p$. Then $d_q = p - 1$ and $d'_q \leq p - 2$.*
(2) *Assume $d'_p \geq p$, and let $q = d'_p + 1$. Then $d_q = p$ and $d'_q \leq p - 1$.*

Proof. Since $S_{n-1}(d) \leq S_{n-1}(d')$ and $S_n(d) = S_n(d')$, we have $d_n \geq d'_n$, so $p < n$ by definition of p. The fact that $p < n$ justifies our mentioning of column $p + 1$ (of C) below. In both cases q is the position of the last 1 in column p, so $C_{qp} = 1$. The assumption on p implies $d'_{p+1} \leq d_{p+1} \leq d_p < d'_p$. Hence $d'_{p+1} < d'_p$, implying that $C_{q,p+1} \neq 1$.

Part 1: See Figure 3.
If $d'_p \leq p - 1$, then $p > q$ and $C_{q,p+1} \neq \star$. It follows that $C_{q,p+1} = 0$, and therefore (since $C_{qq} = \star$ and $C_{qp} = 1$) $d_q = p - 1$. Also $d_p < d'_p = q$ gives $C_{pq} = 0$, and again $C(q, q) = \star$ implies $d'_q \leq p - 2$.

Part 2: In this case, apply Lemma 9 with $i = p$ and $j - 1 = d'_p$. Then $j = d'_p + 1 = q$ and $d'_q < p$ as required. In our case $p < q$. See Figure 4. We have seen that $C_{q,p+1} \neq 1$, but $C_{qp} = 1$, so $d_q \geq p$. If $q > p + 1$, then $C_{q,p+1} = 0$ and $d_q = p$, as required. If $q = p + 1$, then $C_{qp} = C_{p+1,p} = 1$

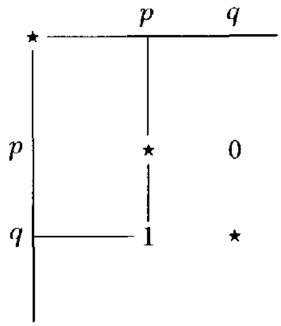

FIG. 4. Illustrating part 2 of Lemma 10.

and $C_{qq} = \star$. On the other hand, $C_{q,q+1}$ must be 0 if it exists, for otherwise $C_{p,p+1}$ would be 1, contradicting $d'_q < p$. Thus again $d_q = p$. ■

LEMMA 11. *Let $d \in D_n$, and let p be as in Lemma 10. For $r = 1, \ldots, p-1$, $d_r \geq p$ implies $d_r \leq d'_r$.*

Proof. Assume to the contrary that $d_r > d'_r$. Let $t = d_r + 1 > p$ (since $d_r > r$, t is the position of the last 1 in row r. See Figure 5.)

By construction $C_{rt} = 1$, and by assumption $C_{tr} = 0$, so $d'_t \geq r$ and $d_t \leq r - 1$. But then $d'_t > d_t$, contradicting the definition of p. ■

LEMMA 12. *Let d be a proper sequence. If there exists a k such that $d'_k < d'_{k+1}$, then*

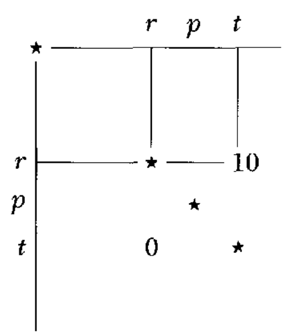

FIG. 5. Illustrating the proof of Lemma 11.

(1) $d'_k = k-1$, $d'_{k+1} = k$;
(2) k *is unique.*

Proof. Part 1: If $d'_k < k-1$ or $d'_k \geq k$, then $d'_{k+1} \leq d'_k$, since the 1's of $C(d)$ are left-justified. Hence $d'_k = k-1$. Then again by this reason and the assumption that $d'_k < d'_{k+1}$, we obtain $d'_{k+1} = k$.

Part 2: The same property of $C(d)$ implies that the conditions $d'_i = i-1$ and $d'_{i+1} = i$ are possible for at most one i. ∎

For a proper sequence d, define

$$m = \max\{k : d_k \geq k-1\}.$$

The parameter m plays an important role throughout our discussion. It is the size of the largest subsquare of $C(d)$ that has one corner at position (1,1) and is full (contains no 0's).

LEMMA 13. (1) *Let $d \in D_n$, and let $C(e)$ be obtained from $C(d)$ by deleting the last* 1 *in row i, where $1 \leq i \leq m$ and*

(a) *if $i < m$, then $d_i > d_{i+1}$;*

(b) *if $i = m$, then $d_m \geq m$.*

Then $e \in D_n$. Further, if $d'_i \geq d_i$, then $\delta(e) > \delta(d)$.

(2) *Let $d \in D_n$, and let $C(e)$ be obtained from $C(d)$ by adding a 1 to the end of column i, where $1 \leq i \leq m$ and*

(a) *if $i > 1$, then $d'_i < d'_{i-1}$;*

(b) *if $i = m$, then $d_m \geq m$.*

Then $e \in D_n$. Further, if $d'_i \geq d_i$, then $\delta(e) > \delta(d)$.

Proof. (1): The assumptions on i guarantee that e is a proper sequence. The column of the 1 that is removed is $j = 1 + d_i > m$. This implies that $e \in D_n$. Further, if $d'_i \geq d_i$, then $C_{ji} = 1$, and thus $d_j \geq i$. But the assumptions on i imply $d'_j = i$. Therefore, by Equation (1),

$$\delta(e) - \delta(d) = (d'_i - d_i + 1)^+ - (d'_i - d_i)^+ + (d'_j - d_j - 1)^+ - (d'_j - d_j)^+ = 1 + 0.$$

(2) is similar. ∎

LEMMA 14. *Let $d \in D_n$ with $d_n > 0$ and $m = 2$. Then $\delta(d) \leq n-2$, with equality if and only if $d_1 = d_2$.*

Proof. Note that the condition $d_n > 0$ implies $n \geq 2$. Using $d_2 \leq d_1 \leq n-1$, $d'_2 = 1$, $d_2 \geq 1$, and $d_i = 1$, $d'_i = 2$, for $i = 3, \ldots, d_2 + 1$, we obtain

$$\delta(d) = \sum_{i=1}^{n} (d'_i - d_i)^+$$

$$
\begin{aligned}
&= (n-1-d_1)^+ + (d_2-1)(2-1) \\
&\leq (n-1-d_2)^+ + d_2 - 1 \\
&= n-1-d_2+d_2-1 \\
&= n-2.
\end{aligned}
$$
■

3. THE MAJORIZATION GAP

We need the next three lemmas to prove Theorem 5.

LEMMA 15. *Let $d \in D_n$. Assume that $d_n > 0$, $d_1 < d'_1$, and let p be the largest i such that $d_i < d'_i$. Let q be such that $d_q > d'_q$ and either (a) $q < n$ or (b) $q = n$, $d_n \geq 2$. Then*

$$S_k(d) < S_k(d') - 1, \qquad \textit{for } k = p, \ldots, q-1.$$

Proof. Since $d \in D_n$, $S_k(d) \leq S_k(d')$. Assume if possible that $S_k(d) \geq S_k(d') - 1$ for some k satisfying $p \leq k \leq q-1$. When $q < n$ we have

$$
\begin{array}{lll}
 & S_k(d) \geq S_k(d') - 1, & \\
\text{(assumption on } p) & d_i \geq d'_i, & i = k+1, \ldots, q-1, \\
\text{(assumption on } q) & d_q \geq d'_q + 1, & \\
\text{(assumption on } p) & d_i \geq d'_i, & i = q+1, \ldots, n-1, \\
(d_1 < d'_1 \Rightarrow d'_n = 0, d_n > 0) & d_n \geq d'_n + 1. &
\end{array}
$$

Adding all these inequalities, we obtain a contradiction, $S_n(d) \geq S_n(d') + 1$.

When $q = n$, the inequality for d_q drops, but by assumption the inequality for d_n can be strengthened by 1, and the same contradiction is obtained. ■

LEMMA 16. *Let $d \in D_n$, assume that $d_1 < d'_1$, $d_n > 0$, and let p be as in Lemma 15. Then*

$$S_k(d) \leq S_k(d') - 1, \qquad k = p, \ldots, n-1.$$

Proof. We have $S_k(d) \leq S_k(d')$, since $d \in D_n$. If $S_k(d) = S_k(d')$, then there exists an i, $k < i < n$, such that $d_i < d'_i$, since $d_n > d'_n$ and $S_n(d) = S_n(d')$. But then $i > p$, contradicting the definition of p. ■

LEMMA 17. *Under the assumptions of Lemma 16:*

(1) *if $d'_p \leq p-1$, then $S_k(d) \leq S_k(d') - 1$ for $k = 1, \ldots, d'_p - 1$;*
(2) *if $d'_p \geq p$, then $S_k(d) \leq S_k(d') - 1$ for $k = 1, \ldots, p-1$.*

Proof. Part 1: Let $q = d'_p \leq p - 1$. By Lemma 10, $d_q = p - 1$ and $d'_q \leq p - 2$, so $d_q > d'_q$. If $d_i \leq d'_i$ for $i = 2, \ldots, q - 1$, then the assumption $d_1 < d'_1$ implies $S_k(d) \leq S_k(d') - 1$ for $k = 1, \ldots, q - 1$, as required. So assume that there exists an i, $2 \leq i \leq q - 1$, such that $d_i > d'_i$, and let r be the smallest such i.

Since $r < q$, $d_r \geq d_q = p - 1$. But the fact that $d_r > d'_r$ and Lemma 11 imply $d_r \leq p - 1$. Therefore $d_r = p - 1$.

We assert that $d_i > d'_i$ for $i = r, \ldots, q$. Indeed, for such i, $p - 1 = d_r \geq d_i \geq d_q = p - 1$, so $d_i = p - 1$. This implies, by the construction of C, that for $i = r, \ldots, q - 1$, $d'_i \geq q - 1 \geq i$, and therefore $d'_i \geq d'_{i+1}$ by Lemma 12. Hence $d'_r \geq d'_{r+1} \geq \cdots \geq d'_{q-1} \geq d'_q$. Thus for $i = r, \ldots, q$, $d'_i \leq d'_r < d_r = p - 1 = d_i$, proving the assertion.

Now for $k = 1, \ldots, r - 1$, the facts $d_1 < d'_1$ and $d_i \leq d'_i$ for $i = 2, \ldots, k$ imply that $S_k(d) \leq S_k(d') - 1$. Also, for $k = r, \ldots, q - 1$, if $S_k(d) = S_k(d')$, then $S_{k+1}(d) > S_{k+1}(d')$ as $d_{k+1} > d'_{k+1}$ by the assertion. But this contradicts $d \in D_n$. Hence again $S_k(d) \leq S_k(d') - 1$.

Part 2: Let $q = d'_p + 1 \geq p + 1$. From Lemma 10, $d_q = p$. Since $p < q$, $d_p \geq d_q = p$. Thus $d_1 \geq d_2 \geq \cdots \geq d_p \geq p$. Using Lemma 11, we obtain $d_i \leq d'_i$ for $i = 1, \ldots, p - 1$. This and the assumption $d_1 < d'_1$ complete the proof. ■

Proof of Theorem 5. From Lemma 8 we have $R(d) \geq \lceil \delta(d)/2 \rceil$, as a reverse unit transformation can decrease $\delta(d)$ by at most 2. We show that if $\delta(d) > 0$, we can always construct a proper sequence e such that

(1) e is obtained from d by a reverse unit transformation,
(2) e is a degree sequence, and
(3) $\delta(e) = \delta(d) - 2$.

It then follows that $\delta(d)$ is even and $R(d) = \delta(d)/2$.

To show that e is a degree sequence we shall use the characterization (3) given in Section 2, i.e.,

$$S_n(e) \text{ is even and } S_k(e) \leq S_k(e') \text{ for } k = 1, \ldots, n.$$

To show that $\delta(e) = \delta(d) - 2$ we shall use the observation made after the proof of Lemma 8.

Without loss of generality, we may assume that $d_n > 0$, for if $d_n = 0$, then we work with $c = (d_1, \ldots, d_{n-1})$. We may also assume that $d_1 < d'_1$, for if $d_1 = d'_1$ $(= n - 1)$, then we work with $c = (d_2 - 1, d_3 - 1, \ldots, d_n - 1)$. We distinguish two cases.

Case 1: *There exists an* $i > 1$ *such that* $d_i < d'_i$. Let p be the largest such i. The basic idea is to transfer the 1 at the end of column p to the end of row 1. Let $s = d_1 + 2$ be the destination column for the moving 1. We have two subcases now.

Subcase 1: $d'_p \leq p - 1$. Then $q = d'_p$ is the source row for the moving 1. Also $d_q = p - 1$ by Lemma 10, and hence p is the source column for the moving 1. Define $e = d - u_q + u_1$. Then for $i = 1, \ldots, n$, (a) $e_i = d_i$ except for $e_1 = d_1 + 1$ and $e_q = d_q - 1$, and (b) $e'_i = d'_i$ except for $e'_p = d'_p - 1$ and $e'_s = 1$. We first prove that e is a degree sequence. Since $S_s(e') = S_n(e)$, it suffices to prove $S_k(e) \leq S_k(e')$ for $k = 1, \ldots, s - 1$. This is done as follows:

$$S_k(e) = S_k(d) + 1 \leq S_k(d') = S_k(e') \text{ for } k = 1, \ldots, q - 1 \text{ (by Lemma 17)},$$
$$S_k(e) = S_k(d) \leq S_k(d') = S_k(e') \quad \text{for } k = q, \ldots, p - 1,$$
$$S_k(e) = S_k(d) \leq S_k(d') - 1 = S_k(e') \text{ for } k = p, \ldots, s - 1 \text{ (by Lemma 16)}.$$

To show that $\delta(e) = \delta(d) - 2$, note that $(\rho, \tau, \pi, \sigma) = (q, p, 1, s)$, where ρ, τ, π, σ are as defined in the proof of Lemma 8. By assumption $p \neq 1$. Also $0 < d_n \leq d_p < d'_p$ implies $p \leq s - 1$, and $q < p$. Hence $q < s - 1$, giving $q \neq s$. Now the necessary and sufficient conditions for $\delta(e) = \delta(d) - 2$ are verified as follows: (a) $d'_1 - d_1 \geq 1$ by assumption; (b) $d'_q - d_q < 0$, as $d_q = p - 1$ and $d'_q \leq p - 2$ by Lemma 10; (c) $d'_s - d_s < 0$, as $d_s \geq d_n \geq 1$ and $d'_s = 0$; and (d) $d'_p - d_p \geq 1$ by definition of p.

Subcase 2: $d'_p \geq p$. From Lemmas 16 and 17 we have

$$S_k(d) < S_k(d'), \qquad k = 1, \ldots, n - 1. \tag{2}$$

Let $q = d'_p + 1$ be the source row for the moving 1, and define e as in subcase 1. We first show that $S_k(e) \leq S_k(e')$ for $k = 1, \ldots, s - 1$. By Lemma 10, $d_q = p$ and $d'_q \leq p - 1$. Note that $s - 1 = d_1 + 1 \leq d'_1 \leq n - 1$. We have:

$$S_k(e) = S_k(d) + 1 \leq S_k(d') = S_k(e') \quad \text{for } k = 1, \ldots, p - 1 \qquad \text{[by (2)]},$$
$$S_p(e) = S_{p-1}(e) + e_p \leq S_{p-1}(e') + e'_p = S_p(e') \text{ (since } e_p = d_p \leq d'_p - 1 = e'_p),$$
$$S_k(e) = S_k(d) + 1 \leq S_k(d') - 1 = S_k(e') \quad \text{for } k = p + 1, \ldots, q - 1 \text{ (by Lemma 15)},$$
$$S_k(e) = S_k(d) \leq S_k(d') - 1 = S_k(e') \quad \text{for } k = q, \ldots, s - 1 \qquad \text{[by (2)]}.$$

To show that $\delta(e) = \delta(d) - 2$, observe that again $(\rho, \tau, \pi, \sigma) = (q, p, 1, s)$. If $q \neq s$, then (a), (c), and (d) are as in subcase 1, and (b) $d'_q - d_q < 0$ by Lemma 10. If $q = s$, then $d'_1 - d_1 \geq 1$ and $d'_p - d_p \geq 1$ as before, and $d'_q - d_q \leq -2$ since $d_q = p \geq 2$ and $d'_q = d'_s = 0$.

Case 2: *$i = 1$ is the only index with the property $d_i < d'_i$.* Then $d_n = 1$, for $d_n \geq 2$ implies $d'_2 = n - 1 = d'_1 > d_1 \geq d_2$, contradicting the assumption of the case.

We assert that $d'_1 \geq d_1+2$. Indeed, we shall show that $d'_1 = d_1+1$ implies that $S_n(d)$ is odd, contradicting the assumption that d is a degree sequence. We have

$$\begin{aligned} &d'_1 = d_1 + 1, \\ &d'_i \leq d_i, \qquad\qquad i = 2, \ldots, n-1, \\ &d'_n = d_n - 1 = 0. \end{aligned}$$

By adding we obtain $S_n(d') \leq S_n(d) = S_n(d')$. Thus all the inequalities above are in fact equalities, i.e., $d'_i = d_i$ for $i = 2, \ldots, n-1$. Therefore the sequence $c = (d_1, d_2, \ldots, d_{n-1}, 0)$ has a symmetric corrected Ferrers diagram. This implies that $S_n(c)$ is even, which in turn implies that $S_n(d)$ is odd. This proves the assertion.

Let $s = d_1+2$, and define $e = d - u_n + u_1$. Then for $i = 1, \ldots, n$, $e_i = d_i$, except for $e_1 = d_1+1$ and $e_n = d_n - 1 = 0$, and $e'_i = d'_i$ except for $e'_1 = d'_1 - 1$ and $e'_s = 1$.

To show that e is a degree sequence, set $q = s$ $(\leq n-1)$, $p = 1$ and apply Lemma 15. Then for $k = 1, \ldots, s-1$, $S_k(e) = S_k(d)+1 \leq S_k(d')-1 = S_k(e')$. Clearly, for $k = s, \ldots, n$, $S_k(e) \leq S_k(e')$.

It remains to verify the necessary and sufficient conditions for $\delta(e) = \delta(d) - 2$. Note that here $(\rho, \tau, \pi, \sigma) = (n, 1, 1, s)$. Observe that $s \neq n$ by the assertion. Now (a) $d'_1 - d_1 \geq 2$ by the assertion; (b) $d'_n - d_n = -1 < 0$; and (c) $d'_s - d_s = -d_s \leq -d_n < 0$. ■

4. THE MAJORIZATION GAP AND THE THRESHOLD GAP

In this section we study the connection between the majorization gap and the threshold gap, introduced by Hammer et al. in [8]. Specifically we show that both gaps are equal and every threshold sequence achieving the majorization gap also achieves the threshold gap.

We first review some results of [8]. For a proper sequence d, define a new sequence $\Delta = (\Delta_1, \ldots, \Delta_n)$, where $\Delta_k = d'_k - d_k$. Recall the definition of m from Section 2. Hammer, Ibaraki and Simeone [8] proved the following result.

LEMMA 18. *For any proper sequence d, $S_n(\Delta) = 0$ and $\sum_{i=1}^{m} |\Delta_i| = \sum_{i=m+1}^{n} |\Delta_i|$.*

We remark that Lemma 18 is especially transparent with the terminology of corrected Ferrers diagrams: both sides of the first equation equal $\sum_{i,j}(C_{ij} - C_{ji})$, and both sides of the second equation equal $\frac{1}{2}\sum_{i,j} |C_{ij} - C_{ji}|$, where $C = C(d)$.

The **threshold gap** of d is defined by $t(d) = \frac{1}{2}\sum_{i=1}^{m} |\Delta_i|$.

Hammer et al. [8] measured the distance between vectors of length n using the norm

$$\|x\| = \frac{1}{2}\sum_{i=1}^{n} |x_i|,$$

which is one-half the L_1-norm. One of their results is that the threshold gap of d is the shortest distance from d to any threshold sequence:

THEOREM 19 ([8]) *For any proper degree sequence d, $t(d) = \min \|d - c\|$, where the minimum is taken over all proper threshold sequences c of the same length as d.*

It turns out that the threshold gap coincides with the majorization gap:

THEOREM 20. *For any proper degree sequence d,*

$$t(d) = R(d).$$

Proof. Let $A = \{i : 1 \le i \le m,\ \Delta_i \ge 0\}$, $A' = \{i : 1 \le i \le m,\ \Delta_i < 0\}$, and $B = \{i : m+1 \le i \le n,\ \Delta_i \ge 0\}$. Define $\alpha_+ = \sum_{i\in A} \Delta_i$, $\alpha_- = \sum_{i\in A'} \Delta_i$, $\beta_+ = \sum_{i\in B} \Delta_i$. It follows from Lemma 18 that $\alpha_- = -\beta_+$. Now

$$\begin{aligned}
t(d) &= \frac{1}{2}\sum_{i=1}^{m} |\Delta_i| \\
&= \frac{1}{2}(\alpha_+ - \alpha_-) \\
&= \frac{1}{2}(\alpha_+ + \beta_+) \\
&= \frac{1}{2}\sum_{i\in A\cup B} (d_i' - d_i) \\
&= \frac{1}{2}\sum_{i=1}^{n} (d_i' - d_i)^+ \\
&= \frac{1}{2}\delta(d) \\
&= R(d) \qquad \text{(by Theorem 5).}
\end{aligned}$$

■

Another main result of Hammer et al. [8] is a characterization of the threshold sequences that achieve $t(d)$. For a proper sequence d, define the sequences $d^+ = (d_1^+, \ldots, d_n^+)$, where $d_k^+ = \max\{d_k, d_k'\}$, and $d^- = (d_1^-, \ldots, d_n^-)$, where $d_k^- = \min(d_k, d_k')$. Note that d^+ and d^- are threshold sequences by Theorem 2.

THEOREM 21. [8] *Let d be a proper degree sequence and c a proper threshold sequence of the same length. Then $\|d - c\| = t(d)$ if and only if $d^- \leq c \leq d^+$.*

The following result states that all threshold sequences achieving $R(d)$ also achieve $t(d)$. Recall that if $a \succcurlyeq b$, $U(a, b)$ denotes the minimum number of successive unit transformations required to transform a into b.

THEOREM 22. *Let d be a proper degree sequence and e a proper threshold sequence such that $e \succcurlyeq d$ and $U(e, d) = R(d)$. Then $\|e - d\| = t(d)$.*

Proof. It suffices to prove $\|e - d\| = R(d)$ by virtue of Theorem 20. Since $e \succcurlyeq d$, we have $S_n(e) = S_n(d)$, and so

$$\sum_{e_i > d_i} (e_i - d_i) = \sum_{e_i < d_i} (d_i - e_i). \tag{3}$$

We then have, where n is the length of d and e,

$$\begin{aligned}
\|e - d\| &= \frac{1}{2} \sum_{i=1}^{n} |e_i - d_i| \\
&= \frac{1}{2} \sum_{e_i > d_i} (e_i - d_i) + \frac{1}{2} \sum_{e_i < d_i} (d_i - e_i) \\
&= \sum_{e_i > d_i} (e_i - d_i) \qquad \text{[by (3)]} \\
&= \sum_{i=1}^{n} (e_i - d_i)^+ \\
&= \delta(e, d) \\
&= U(e, d) \qquad \text{(by Lemma 7)} \\
&= R(d).
\end{aligned}$$

■

REMARK 23. The assumptions that e is threshold, $\|e - d\| = t(d)$, and $S_n(d) = S_n(e)$ do not imply that $e \succcurlyeq d$. For example, consider $d = (6, 6, 4, 4, 3, 3, 1, 1)$ and $e = (7, 6, 4, 3, 3, 2, 2, 1)$.

5. MAXIMUM MAJORIZATION GAP

The problem of characterizing the degree sequences with maximum majorization gap is discussed in this section. We show that for a fixed number of edges and a variable number of vertices, $R(d)$ is maximized precisely when d is the degree sequence of a matching plus isolated vertices. For a fixed number n of vertices and a variable number of edges, we exhibit all the degree sequences that maximize $R(d)$, and they turn out to be almost $n/2$-regular.

As is customary, we denote the constant sequence $(a, a, \ldots, a)$ of length n by a^n, the sequence $(a, b, \ldots, b)$ by ab^{n-1}, etc.

THEOREM 24. *Let d be a proper degree sequence with a fixed sum $2q > 0$. Then*

$$R(d) \leq q - 1.$$

Furthermore, equality holds if and only if d is of the form $d = 1^{2q}0^r$.

Proof. We have

$$\begin{aligned}
\delta(d) &= \sum_{i\geq 1}(d_i' - d_i)^+ \\
&= (d_1' - d_1)^+ + \sum_{i\geq 2}(d_i' - d_i)^+ \\
&\leq d_1' - d_1 + \sum_{i\geq 2} d_i' \qquad \text{(since } d_1' \geq d_1 \text{ and } d_i, d_i' \geq 0).
\end{aligned}$$

If $d_1 \geq 2$, then $\delta(d) \leq \sum_{i\geq 1} d_i' - 2 = 2q - 2$. If $d_1 = 1$, then $d = 1^{2q}0^r$, and so $\delta(d) = 2q - 2$.

Conversely, let d satisfy $\sum_{i\geq 1}(d_i' - d_i)^+ = 2q - 2$. Assume if possible $d_1 > 1$. Then

$$\begin{aligned}
(d_1' - d_1)^+ &= d_1' - d_1, \\
2 &\leq d_1, \\
(d_i' - d_i)^+ &\leq d_i', \qquad i \geq 2.
\end{aligned}$$

Adding these inequalities, we obtain $2q = 2q$; hence all the above inequalities hold as equalities. Therefore for $i \geq 2$, if $d_i' > 0$ then $d_i = 0$. But this fails for $i = 2$, as $d_1' \geq d_1 = 2$. ∎

In the rest of this section we consider the majorization gap of the degree sequences of a fixed length n. We shall make use of the following functions:

$$\begin{aligned}
f(n) &= \lfloor \tfrac{n-1}{2} \rfloor \lceil \tfrac{n-1}{2} \rceil, \\
g(n) &= \begin{cases} f(n) - 1 & \text{if } n \equiv 3 \bmod 4, \\ f(n) & \text{otherwise.} \end{cases}
\end{aligned}$$

We reproduce the definition of D_n for convenience. For positive integers n,

$$D_n = \{d = (d_1, \ldots, d_n) : d \text{ is proper, } S_k(d) \leq S_k(d') \text{ for } k = 1, \ldots, n\}.$$

THEOREM 25. *Let d be a proper degree sequence of length n. Then $\delta(d) \leq g(n)$. Further, for $n \geq 5$, equality holds if and only if*

$$\begin{array}{lll} \textit{for } n \not\equiv 3 \bmod 4 & d = \lceil \frac{n-1}{2} \rceil^n \quad \textit{or} \quad d = \lfloor \frac{n-1}{2} \rfloor^n; \\ \textit{for } n \equiv 3 \bmod 4 & d = \frac{n+1}{2}\left(\frac{n-1}{2}\right)^{n-1}, \quad \textit{or} \quad d = \left(\frac{n-1}{2}\right)^{n-1}\frac{n-3}{2}, \\ & \textit{or} \quad d = \left(\frac{n-3}{2}\right)^n, \quad \textit{or} \quad d = \left(\frac{n+1}{2}\right)^n. \end{array}$$

To prove Theorem 25, we find it convenient to prove the following theorem:

THEOREM 26. *For $d \in D_n$, $\delta(d) \leq f(n)$. Further, for $n \geq 5$, equality holds if and only if $d = \lceil (n-1)/2 \rceil^n$ or $d = \lfloor (n-1)/2 \rfloor^n$.*

We need several results to prove these theorems. First, some simple facts about the functions $f(n)$ and $g(n)$:

(1) $f(n-1) \leq f(n)$ for $n \geq 1$; $f(n-1) < f(n)$ for $n \geq 3$;
(2) $f(n) \geq n-2$ for $n \geq 2$, with strict inequality for $n \geq 5$;
(3) $g(n-1) < g(n)$ for $n \geq 4$; $g(n) \geq n+1$ for $n \geq 7$.

Next, three lemmas about corrected Ferrers diagrams.

LEMMA 27. *Let $0 \neq d \in D_n$. Put $s = d_1 + 1$, $p = d'_s$, $q = d'_p + 1$, $r = d_s$. Thus $s \geq m \geq p \geq 1$ and $q \geq m \geq r \geq 1$. Assume that the following hold (See Figure 6):*

(1) *$s \geq m+1$, and if equality holds, then $p < m$;*
(2) *$q \leq s$;*
(3) *$r \leq p - r$.*

Then:

(1) *there exists a sequence $e \in D_n$ such that $e_1 = d_1 - 1$ and $\delta(e) \geq \delta(d)+2$;*
(2) *there exists a sequence $f \in D_n$ such that*
   (a) *(i) $f_1 = d_1 - 1$ or (ii) $f_1 = d_1$ and $f'_s = 1$;*
   (b) *$\delta(f) \geq \delta(d) + 1$;*
   (c) *$S_n(f)$ and $S_n(d)$ have the same parity.*

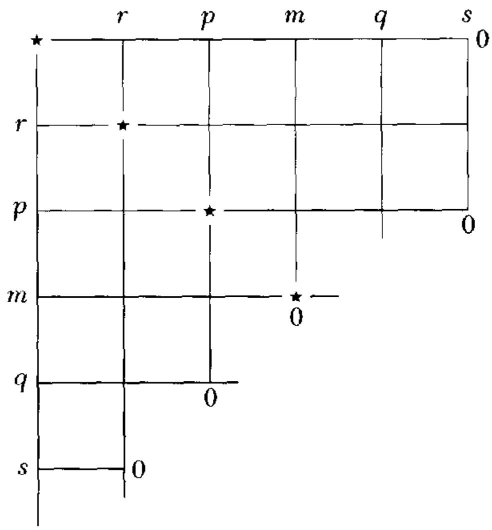

FIG. 6. Illustrating Lemma 27.

Proof. We begin by proving statement 1. First note that $p \le m \le q$ and $r < p$. Secondly, we may assume that the first r columns of $C(d)$ are full, i.e.,

$$d'_i = n - 1, \qquad i = 1, \ldots, r, \tag{4}$$

for otherwise we can fill column 1, then 2, and so on up to r (by adding 1's at the end of these columns) without going out of D_n, and increase $\delta(d)$ at each step, by Lemma 13.

Counting in two ways the number of 1's in rows $1, \ldots, s$ and columns $r+1, \ldots, p$ of $C(d)$, we obtain

$$\sum_{i=r+1}^{p} d'_i = (p-r)(q-1) + \sum_{i=q+1}^{s} (d_i - r). \tag{5}$$

Since $d \in D_n$, we have $S_p(d) \le S_p(d')$. This implies, by (4) and (5),

$$p(s-1) \le r(n-1) + \sum_{i=r+1}^{p} d'_i = r(n-1) + (p-r)(q-1) + \sum_{i=q+1}^{s} (d_i - r).$$

Therefore $p(s-q) \le r(n-1-q) + r + \sum_{i=q+1}^{s}(d_i - r) < r(n-1-q) + p + \sum_{i=q+1}^{s}(d_i - r)$, and it follows that

$$(p-r)(s-q) < r(n-1-s) + p + \sum_{i=q+1}^{s}(d_i - r). \tag{6}$$

By Lemma 12, $d'_{r+1} \ge d'_{r+2} \ge \cdots \ge d'_p$, for if $d'_i < d'_{i+1}$ for some $r < i < p$, then $q - 1 \le d'_i < d'_{i+1} = i < p \le q$, a contradiction. Also $d_i = s - 1$ for $i = r+1, \ldots, p$, and $d'_{r+1} \le s - 2$. Hence

$$d_i > d'_i, \qquad i = r+1, \ldots, p. \tag{7}$$

Again by Lemma 12, $d'_{q+1} \ge d'_{q+2} \ge \cdots \ge d'_s$, for if $d'_i < d'_{i+1}$ for some $q < i < s$, then $d'_{i+1} = i > q \ge m$, a contradiction. Therefore for $i = q+1, \ldots, s$, we have $d_i \le d_{q+1} \le p - 1$ and $d'_i \ge d'_s = p$. Hence

$$d'_i > d_i, \qquad i = q+1, \ldots, s. \tag{8}$$

Using (4), (7), and (8), we have

$$\delta(d) = r(n-s) + \sum_{i=p+1}^{q}(d'_i - d_i)^+ + \sum_{i=q+1}^{s}(d'_i - d_i). \tag{9}$$

Define a sequence e such that $C(e)$ is obtained from $C(d)$ by making the first p columns full and deleting the sth column, i.e.,

$$e'_i = \begin{cases} n-1 & \text{for } i = 1, \ldots, p, \\ 0 & \text{for } i = s, \\ d'_i & \text{otherwise.} \end{cases}$$

It is easy to check (using $s \ge m+1$) that $e \in D_n$ by Lemma 13. Further,

$$\begin{aligned}
\delta(e) &= \sum_{i=1}^{s-1}(e'_i - e_i)^+ \\
&= p(n-s+1) + \sum_{i=p+1}^{q}(d'_i - d_i)^+ + \sum_{i=q+1}^{s-1}(d'_i - p) \\
&= r(n-s) + (p-r)(n-s) + p + \sum_{i=p+1}^{q}(d'_i - d_i)^+ + \sum_{i=q+1}^{s}(d'_i - p) \\
&\qquad (\text{as } d'_s = p)
\end{aligned}$$

$$= r(n-s) + (p-r)(n-s) + p + \sum_{i=p+1}^{q} (d_i' - d_i)^+$$

$$+ \sum_{i=q+1}^{s} (d_i' - d_i) - \sum_{i=q+1}^{s} (p - d_i)$$

$$= \delta(d) + (p-r)(n-s) + p - \sum_{i=q+1}^{s} (p - d_i) \qquad \text{[by (9)]}$$

$$= \delta(d) + (p-r)(n-s) + p - \sum_{i=q+1}^{s} (p - r) + \sum_{i=q+1}^{s} (d_i - r)$$

$$= \delta(d) + (p-r)(n-s) + p - (s-q)(p-r) + \sum_{i=q+1}^{s} (d_i - r)$$

$$> \delta(d) + (p-r)(n-s) - r(n-1-s) \qquad \text{[by (6)]}$$

$$= \delta(d) + (p-r)(n-s) - r(n-s) + r$$

$$\geq \delta(d) + r \qquad (\text{ as } r \leq p - r).$$

So $\delta(e) \geq \delta(d) + 2$. This proves statement 1.

We now prove statement 2. If $S_n(e)$ has the same parity as $S_n(d)$ [before the achievement of (4)], then $f = e$ has the required properties. If the parities differ, take $f = e + u_1$. In this case $C(f)$ can be obtained from $C(d)$ by making the first p columns full and deleting all the 1's except the first 1 in the sth column. So $f \in D_n$ by Lemma 13. The other required properties of e follow from $S_n(f) = S_n(e) + 1$, $\delta_1(f) = \delta_1(e) - 1$, and $\delta_i(f) = \delta_i(e)$ for $i = 2, \ldots, n$. ■

Lemma 28 relaxes one of the assumptions of Lemma 27.

LEMMA 28. *Under the conditions of Lemma 27 except $r \leq p - r$:*

(1) *there exists a sequence $e \in D_n$ such that $e_1 = d_1 - 1$ and $\delta(e) \geq \delta(d) + 1$;*
(2) *there exists a sequence $f \in D_n$ such that*

(a) (i) $f_1 = d_1 - 1$ *or* (ii) $f_1 = d_1$ *and* $f_s' = 1$;
(b) *in case* (i) *above,* $\delta(f) \geq \delta(d) + 1$; *in case* (ii) *above,* $\delta(f) \geq \delta(d)$;
(c) $S_n(f)$ *and* $S_n(d)$ *have the same parity.*

Proof. If $r \leq p - r$, then the results follow from Lemma 27, so assume $r > p - r$. As in the proof of Lemma 27, we may assume that (4) holds. We also deduce equation (9) as before. Define a sequence e such that $C(e)$ is obtained from $C(d)$ by deleting the sth column. Then $e \in D_n$ by Lemma 13, and

$$\delta(e) = \sum_{i=1}^{s-1} (e_i' - e_i)^+$$

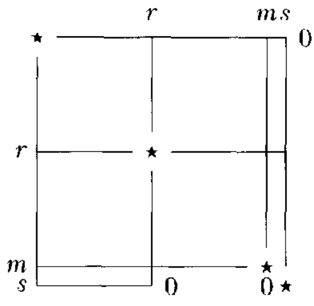

FIG. 7. Illustrating Lemma 29.

$$= r(n+1-s) + \sum_{i=p+1}^{q} (d'_i - d_i)^+ + \sum_{i=q+1}^{s-1} (d'_i - d_i)$$
$$= r(n-s) + r + \sum_{i=p+1}^{q} (d'_i - d_i)^+ + \sum_{i=q+1}^{s} (d'_i - d_i) - d'_s + d_s$$
$$= \delta(d) + r - p + r \qquad \text{[by (9) and the definition of } p \text{ and } r]$$
$$> \delta(d) \qquad (\text{as } r > p - r).$$

This proves statement 1. To prove statement 2, define f from e as in the proof of Lemma 27. ■

Lemma 29 is a variation of Lemma 27.

LEMMA 29. *Let $0 \neq d \in D_n$. Put $s = d_1 + 1$, $p = d'_s$, $r = d_s$. Assume that $s = m + 1$ and $p = m$ (see Figure 7).*
Then

(1) *if $m \geq 3$, then there exists a sequence $e \in D_n$ such that $e_1 = d_1 - 1$ and $\delta(e) \geq \delta(d) + 1$;*
(2) *if $m \geq 4$, then there exists a sequence $f \in D_n$ such that*
   (a) *(i) $f_1 = d_1 - 1$ or (ii) $f_1 = d_1$ and $f'_s = 1$;*
   (b) *$\delta(f) \geq \delta(d) + 1$;*
   (c) *$S_n(f)$ and $S_n(d)$ have the same parity.*

Proof. We begin by proving statement 1. First, observe that $m < n - 1$, because $m = n - 1$ and $p = m$ would imply $S_m(d) > S_m(d')$, contradicting $d \in D_n$. Secondly, assume without loss of generality that (4) holds as in the proof of Lemma 27. Define a sequence e such that $C(e)$ is obtained from $C(d)$ by making the first $m - 1$ columns full, if necessary, and deleting the column $s = m + 1$, i.e.,

$$e_i' = \begin{cases} n-1 & \text{for } i = 1, \ldots, m-1, \\ 0 & \text{for } i = m+1, \\ d_i' & \text{otherwise.} \end{cases}$$

Then $e \in D_n$ by Lemma 13. Using the facts $d_i' = n - 1$ for $i = 1, \ldots, r$, $d_i = m$, $d_i' = m - 1$ for $i = r + 1, \ldots, m$, $d_{m+1}' = m$, $d_{m+1} = r \leq m - 1$ (by definition of m and r), we obtain

$$\delta(d) = r(n - 1 - m) + m - r \leq (m - 1)(n - 1 - m) + m - r.$$

Also,

$$\delta(e) = (m - 1)[n - 1 - (m - 1)] = (m - 1)(n - 1 - m) + m - 1.$$

It is now clear that if $r > 1$, then $\delta(e) > \delta(d)$. If $r = 1$, the same conclusion holds, since $m - 1 > 1$ and $n - 1 - m > 0$. This proves statement 1.

We now prove statement 2. If $S_n(d)$ before the achievement of (4) and $S_n(e)$ have the same parity, then $f = e$ has the required properties. Otherwise take $f = e + u_{m+1}$. Once again, $C(f)$ can be obtained from $C(d)$ by making the first $m - 1$ columns full and deleting all the 1's except the first 1 in column $m + 1$. Therefore $f \in D_n$ by Lemma 13. Further, $\delta(f) = \delta(e) - 1$, as $\delta_1(f) = \delta_1(e) - 1$, and $\delta_i(f) = \delta_i(e)$ for $i = 2, \ldots, m$. Hence

$$\delta(f) = (m - 1)(n - 1 - m) + m - 2.$$

Now it is clear that if $r > 2$, then $\delta(f) > \delta(d)$. If $r = 2$, the same conclusion holds since $m - 1 > r$ (because $m \geq 4$) and $n - 1 - m > 0$. Finally, if $r = 1$, then the conclusion holds again, since

$$\begin{aligned} \delta(f) &= (m - 2)(n - 1 - m) + n - 1 - m + m - 2 \\ &\geq (m - 2)(n - 1 - m) + m - 1 \quad (\text{as } n - 2 \geq m) \\ &> n - 1 - m + m - 1 \quad (\text{as } m \geq 4 \text{ and } n - 1 - m > 0) \\ &= \delta(d). \end{aligned}$$

■

We are now ready to prove Theorems 25 and 26.

Proof of Theorem 26. The statement is true for $n = 1$, so assume $n \geq 2$. If $d_n = 0$, then by induction on n, $\delta(d) \leq f(n-1) \leq f(n)$, and $f(n-1) < f(n)$ for $n \geq 3$. We may therefore assume that $d_n > 0$, and consequently $m \geq 2$. If $m = 2$, then $\delta(d) \leq n - 2$ by Lemma 14, and hence $\delta(d) \leq f(n)$. Equality holds if and only if $d_1 = d_2$ and $n - 2 = f(n)$, which implies $n \leq 4$. Hence we may assume $m \geq 3$. Put $s = d_1 + 1$, $p = d'_s$, $q = d'_p + 1$, $r = d_s$, and observe as before that $p \leq m \leq q$ and $m \leq s$.

PROPOSITION. *If $s \geq m + 1$, then there exists a sequence $e \in D_n$ such that $e_1 = d_1 - 1$ and $\delta(e) > \delta(d)$. Consequently we may assume that $s = m$.*

To prove the proposition, first assume that $q \geq s$. Define a sequence e such that $C(e)$ is obtained from $C(d)$ by making the first p columns full and deleting the sth column. Then $e \in D_n$ and $\delta(e) > \delta(d)$ by Lemma 13. Now assume $q < s$. Then $r = d_s \leq d_{q+1} \leq p - 1$, and hence $r < p$. If $s > m + 1$ or $p < m$, then the required sequence e exists by Lemma 28. If $s = m + 1$ and $p = m$, then the required sequence e exists by Lemma 29.

This proves the proposition, and we may assume that $s = m$. Further, we may assume that the first $m - 1$ columns of d are full, for otherwise we make them full without leaving D_n, thereby increasing $\delta(d)$ by Lemma 13. Then

$$d = (m-1)^n,$$
$$\delta(d) = (m-1)[n-1-(m-1)],$$

and $\delta(d)$ reaches a maximum when

$$m - 1 = \begin{cases} \frac{n-1}{2}, & n \text{ odd}, \\ \frac{n}{2} - 1, \frac{n}{2}, & n \text{ even}. \end{cases} \tag{10}$$

Therefore

$$\delta(d) \leq \begin{cases} (\frac{n-1}{2})^2, & n \text{ odd}, \\ \frac{n}{2}(\frac{n}{2} - 1), & n \text{ even}. \end{cases}$$

This means that $\delta(d) \leq f(n)$. Further, for $n \geq 5$, equality holds if and only if $d = (m-1)^n$, where $m - 1$ is given by equation (10), i.e., if and only if

$$d = \left\lceil \frac{n-1}{2} \right\rceil^n \quad \text{or} \quad d = \left\lfloor \frac{n-1}{2} \right\rfloor^n. \qquad \blacksquare$$

Proof of Theorem 25. We use the notation

$$E_n = \{d \in D_n : S_n(d) \text{ even}\},$$

i.e., $E_n(d)$ is the set of all proper degree sequences of length n. Since $E_n \subseteq D_n$, we have $\delta(d) \leq f(n)$ by Theorem 26. For $n \not\equiv 3 \bmod 4$, $f(n) = g(n)$ and hence $\delta(d) \leq g(n)$. The cases of equality for $d \in D_n$ are when $d = \lceil (n-1)/2 \rceil^n$ or $d = \lfloor (n-1)/2 \rfloor^n$. Since these d belong to E_n when $n \not\equiv 3 \bmod 4$, all the conclusions of Theorem 25 are established in this case. However, for $n \equiv 3 \bmod 4, f(n) = [(n-1)/2]^2$ is odd, whereas $\delta(d)$ is even by Theorem 5, since d is a degree sequence. Therefore $\delta(d) \leq f(n) - 1 = g(n)$ for $n \equiv 3 \bmod 4$. It remains to track down the cases of equality for $n \equiv 3 \bmod 4$. Thus from now on, we assume that $n \equiv 3 \bmod 4$, $n \geq 7$, and $\delta(d) = g(n)$. If $d_n = 0$, then $\delta(d) \leq g(n-1) < g(n)$, contradicting our assumption, so we may assume that $d_n > 0$, and therefore $m \geq 2$. Also $m = 2$ implies, by Lemma 14, that $\delta(d) \leq n - 2 < g(n)$, again contradicting our assumption, so we assume that $m \geq 3$. Put $s = d_1 + 1$.

PROPOSITION. *If $s \geq m + 2$, then there exists a sequence $f \in E_n$ such that $\delta(f) > \delta(d)$, contradicting $\delta(d) = g(n)$. Consequently we may assume that $s = m$ or $s = m + 1$.*

We prove the proposition by showing that, when $s \geq m + 2$,

(1) if $d'_s = 1$, then there exists an $f \in E_n$ such that $\delta(f) > \delta(d)$;
(2) if $d'_s \geq 2$, then there exists an $f \in E_n$ such that $\delta(f) \geq \delta(d)$, and if equality holds, then $f_1 = d_1$ (so that f and d have the same s) and $f'_s = 1$.

Case 1: $d'_s = 1$. The basic idea is to work with the sequence $(d_1 - 1, d_2, \ldots, d_n)$ and introduce a 1 at the end of the first row if the parity becomes odd. Put $t = s - 1$, $p = d'_t$, $q = d'_p + 1$. Then $p \leq m \leq q$. Assume that $q \geq t$. Define a sequence f such that $C(f)$ is obtained from $C(d)$ by deleting the last 1 in column $s - 1$ and the 1 in column s. Then $f \in E_n$ and $\delta(f) > \delta(d)$ by Lemma 13. Therefore we may assume $q < t$. Put $r = d_t$. See Figure 8.

Subcase 1.1: $t > m+1$ or $p < m$. Define a sequence c such that $C(c)$ is obtained from $C(d)$ by deleting the last 1 in the first row. Note that $c \in D_n$ by Lemma 13 and, further, c satisfies all the hypotheses of Lemma 28. By the latter, there exists a sequence $g \in D_n$ such that $\delta(g) \geq \delta(c) + 1$. The required sequence f is defined by

$$f = \begin{cases} g, & S_n(g) \text{ even}, \\ (g_1 + 1, g_2, \ldots, g_n), & S_n(g) \text{ odd}. \end{cases}$$

To see this, recall that $C(g)$ is obtained from $C(c)$ in the proof of Lemma 28 by making the first p columns full and deleting the tth column (if $r \leq p - r$), or by making the first r columns full and deleting the tth column (if $r > p-r$). Note that when $S_n(g)$ is odd, $C(f)$ could be obtained from $C(c)$ by making the appropriate columns full and then deleting all but the first 1 in column t. Therefore $f \in D_n$ in this case by Lemma 13. Clearly $f \in D_n$ also holds when $S_n(g)$ is even. Further, $S_n(f)$ is even in both cases. Thus $f \in E_n$.

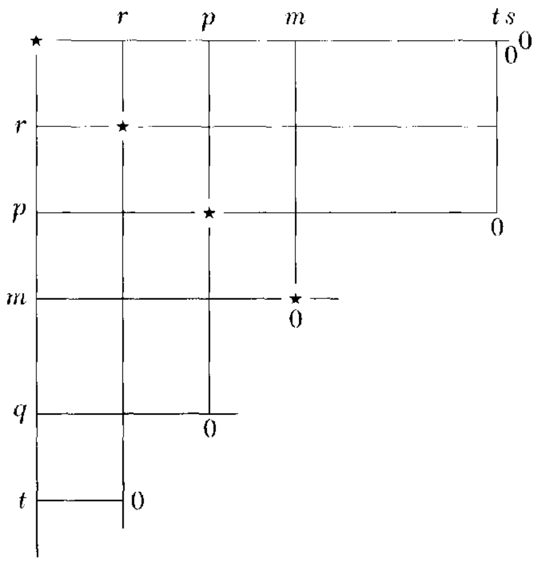

FIG. 8. Illustrating case 1 of the proposition in the proof of Theorem 25.

From the construction of c and f it is clear that $\delta(c) = \delta(d) + 1$ and $\delta(f) \geq \delta(g) - 1$. Therefore $\delta(f) \geq \delta(g) - 1 \geq \delta(c) = \delta(d) + 1$.

Subcase 1.2: $t = m + 1$ and $p = m$. This case is similar to subcase 1.1, except that here we use Lemma 29 instead of Lemma 28.

Case 2: $2 \leq d'_s \leq m$. Put $p = d'_s$ and $q = d'_p + 1$. If $q \geq s$, then we may remove the last two 1's in the s-th column without leaving E_n and thereby increase $\delta(d)$ by Lemma 13. We therefore assume that $q < s$. Then the required f exists by Lemma 28.

This completes the proof of the proposition and hence we may assume that $s = m$ or $s = m + 1$.

It is convenient here to define a new function. Let $\alpha \geq 6$ be a fixed integer such that $\alpha \equiv 2 \bmod 4$. For $0 \leq k \leq \alpha$, define

$$h_\alpha(k) = \begin{cases} k(\alpha - k), & k \text{ even}, \\ k(\alpha - k) - 1, & k \text{ odd}, \end{cases}$$

i.e.,

$$h_\alpha(k) = 2\left\lfloor \frac{k(\alpha - k)}{2} \right\rfloor.$$

Note that the maximum of $h_\alpha(k)$ occurs at $k = \alpha/2 - 1$, $\alpha/2$, $\alpha/2 + 1$, and the maximum is $(\alpha^2 - 4)/4$.

PROPOSITION. *If $s = m$ or $s = m+1$, then $\delta(d) \le h_{n-1}(m-1)$, with equality if and only if*

$$d = (m-1)^n \qquad \text{for } m \text{ odd},$$
$$d = (m-1)^{n-1}(m-2) \quad \text{or} \quad d = m(m-1)^{n-1} \qquad \text{for } m \text{ even}.$$

To prove the proposition, consider first the case that $s = m$. If columns $1, \ldots, m-1$ are made full in that order, then d stays in D_n and $\delta(d)$ increases at each step by Lemma 13. For odd m, the resulting $d = (m-1)^n$ also belongs to E_n, and so it is the only sequence of E_n satisfying $s = m$ that maximizes $\delta(d)$. The maximum in this case equals $(m-1)[n-1-(m-1)] = h_{n-1}(m-1)$. For even m, the above d does not belong to E_n, and so the only sequence of E_n satisfying $s = m$ that maximizes $\delta(d)$ is the previous sequence in the filling-up process, namely $d = (m-1)^{n-1}(m-2)$. The maximum in this case equals $(m-1)[n-1-(m-1)] - 1 = h_{n-1}(m-1)$.

Now consider the case that $s = m+1$. We assert that if $d'_s \ge 2$, then there exists $f \in E_n$ such that $\delta(f) > \delta(d)$, and consequently we may assume that $d'_s = 1$. To prove the assertion we distinguish two cases.

Case 1: $2 \le d'_s < m$. Put $p = d'_s$, $r = d_s$, $q = d'_p + 1$. We may assume that $q < s$, for otherwise we may remove the last two 1's in the sth column of d without leaving E_n, thereby increasing $\delta(d)$ by Lemma 13. Thus $q = m$. If $r \le p - r$, then the required f exists by Lemma 27, so we assume that $r > p - r$. Using $d'_i \ge m$ and $d_i = m$ for $i = 1, \ldots, r$, $d'_i = m-1$ and $d_i = m$ for $i = r+1, \ldots, p$, $d'_i = d_i = m-1$ for $i = p+1, \ldots, m$, $d'_{m+1} = p$, $d_{m+1} = r$, and $d'_i = 0$ for $i = m+2, \ldots, n$, we obtain

$$\delta(d) = \sum_{i=1}^{r}(d'_i - d_i) + p - r. \tag{11}$$

Let f be a sequence such that $C(f)$ is obtained from $C(d)$ by (a) deleting the sth column and (b) adding a 1 at the end of the $(r+1)$st column if p is odd. Then $f \in E_n$ by Lemma 13 and

$$\begin{aligned}
\delta(f) &\ge \sum_{i=1}^{r}[d'_i - (d_i - 1)] \\
&= \sum_{i=1}^{r}(d'_i - d_i) + r \\
&> \delta(d) \qquad [\text{by (11) and } r > p - r].
\end{aligned}$$

$$
\begin{array}{r|cccc}
 & & & m & s \\
 & \star & 1 & 1 & 0 \\
 & 1 & \star & 1 & 1 \\
m & 1 & 1 & \star & 1 \\
s & 1 & 1 & 0 & \\
 & \vdots & \vdots & & \\
d'_2+1 & 1 & 1 & & \\
 & 1 & 0 & & \\
 & \vdots & & & \\
n & 1 & & &
\end{array}
$$

FIG. 9. Illustrating a special case in the proof of Theorem 25.

Case 2: $d'_s = m$. If $m \geq 4$, then the required f exists by Lemma 29. Now assume the special case $m = 3$. Then $d = 3^3 2^{d'_2-2} 1^{n-d'_2-1}$ with $d'_2 \geq 2$ (see Figure 9).

If $d'_2 = 2$, then $S_n(d) = n + 6$ is odd, since $n \equiv 3 \bmod 4$, contradicting $d \in E_n$. Therefore $d'_2 \geq 3$. Hence $\delta(d) = (n-4)^+ + (d'_2-3)^+ + (2-3)^+ + (3-2)^+ = n + d'_2 - 6 \leq 2n - 7$, and writing $n = 4k+3$, we have $\delta(d) \leq 8k - 1 < 4k^2 + 4k = (2k+1)^2 - 1 = g(n)$, contradicting the assumption $\delta(d) = g(n)$.

This completes the proof of the assertion, and consequently we may assume that $d'_s = 1$.

If m is odd, then make the first $m-1$ columns of $C(d)$ full and delete the last 1 in the first row without leaving E_n, thereby increasing $\delta(d)$ by Lemma 13. Therefore m may be assumed to be even. Then it is easy to check as before that $d = m(m-1)^{n-1}$ is the only sequence in E_n satisfying $s = m+1$ and $d'_s = 1$ that maximizes $\delta(d)$, and $\delta(d) = h_{n-1}(m-1)$.

This proves the proposition, and so $\delta(d) \leq h_{n-1}(m-1)$. The function $h_{n-1}(m-1)$ reaches its maximum when

$$m - 1 = \tfrac{n-1}{2} - 1, \quad \text{or} \quad m - 1 = \tfrac{n-1}{2}, \quad \text{or} \quad m - 1 = \tfrac{n-1}{2} + 1.$$

The maximum is $[(n-1)/2]^2 - 1 = g(n)$. By the previous proposition it is achieved only by the following sequences d:

when $m - 1 = \frac{n-1}{2} - 1$ (m odd): $\quad d = \left(\frac{n-3}{2}\right)^n$;

when $m - 1 = \frac{n-1}{2}$ (m even): $\quad d = \left(\frac{n-1}{2}\right)^{n-1} \frac{n-3}{2}$ or $d = \frac{n+1}{2} \left(\frac{n-1}{2}\right)^{n-1}$;

when $m - 1 = \frac{n-1}{2} + 1$ (m odd): $\quad d = \left(\frac{n+1}{2}\right)^n$. ■

6. THE DIFFERENCE GAP

This section considers the bipartite analog of the majorization gap. A graph $G = (V, E)$ with $V = \{1, \ldots, n\}$ is said to be a **difference graph** (also known as a chain graph) if there exist real numbers $a_1, a_2, \ldots, a_n$ and a positive real number T such that

(1) $|a_i| < T$ for $i = 1, 2, \ldots, n$;
(2) distinct vertices i and j are adjacent if and only if $|a_i - a_j| \geq T$.

Such a difference graph is bipartite, with the bipartition $V = X \cup Y$, where $X = \{i : a_i \geq 0\}$, $Y = \{i : a_i < 0\}$. Several characterizations of difference graphs and polyhedral properties of their degree sequences are reported in [10]. In particular, they are bipartite analogs of threshold graphs in the following sense: If all the edges between the vertices of X are added to a difference graph to make X a clique, the resulting graph is a threshold graph; conversely, every threshold graph can be obtained in this way.

A pair of integer sequences $d = (d_1, d_2, \ldots, d_p)$ and $f = (f_1, f_2, \ldots, f_q)$, where $p+q = n$, is called a **bipartite** (respectively, a **difference**) **sequence** if there exists a bipartite (a difference) graph on the vertex set $V = \{1, 2, \ldots, n\}$ with bipartition $\{1, 2, \ldots, p\} \cup \{p + 1, \ldots, n\}$ such that $\deg(i) = d_i$ for $i = 1, \ldots, p$ and $\deg(i) = f_{i-p}$ for $i = p + 1, \ldots, n$.

If $(f_1, f_2, \ldots, f_p)$ is a nonnegative integer sequence, its **conjugate sequence** is the sequence $(f_1^*, f_2^*, \ldots)$, where

$$f_k^* = \left|\{i : f_i \geq k\}\right|.$$

The following theorem is known as the Gale-Ryser theorem [5, 16]. It is perhaps the earliest result that relates degree sequences and majorization. See [4, 17, 1, 14] for further details. This theorem is the bipartite analog of the result that a proper sequence d is a degree sequence if and only if $S_n(d)$ is even and $S_k(d) \leq S_k(d')$ for all k.

THEOREM 30. *Let $d = (d_1, d_2, \ldots, d_p)$ and $f = (f_1, f_2, \ldots, f_q)$ be a pair of nonnegative integer sequences, where $p \geq f_1 \geq f_2 \geq \cdots \geq f_q$. Then (d, f) is a bipartite sequence if and only if*

$$(d_1, d_2, \ldots, d_p) \preccurlyeq (f_1^*, f_2^*, \ldots, f_p^*).$$

The following result is the bipartite analog of the result that a proper sequence d is a threshold sequence if and only if $d = d'$.

THEOREM 31. [10] *Let* $d = (d_1, d_2, \ldots, d_p)$ *and* $f = (f_1, f_2, \ldots, f_q)$ *be a pair of nonnegative integer sequences, where* $p \geq f_1 \geq f_2 \geq \cdots \geq f_q$ *and* $d_1 \geq d_2 \geq \cdots \geq d_p$. *Then* (d, f) *is a difference sequence if and only if*

$$(d_1, d_2, \ldots, d_p) = (f_1^*, f_2^*, \ldots, f_p^*).$$

The following result is the bipartite analog of the result that a degree sequence d is threshold if and only if there does not exist a degree sequence e such that $d \prec e$.

THEOREM 32. [10] *A bipartite sequence* (d, f) *is a difference sequence if and only if there does not exist a bipartite sequence* (e, f) *such that* $d \prec e$.

Motivated by Theorem 32, we propose the following definition: the **difference gap** of a bipartite sequence (d, f), denoted by $R(d, f)$, is the minimum number of successive reverse unit transformations required to transform d into a sequence e such that (e, f) is a difference sequence. In analogy with Theorem 5, we have the following result:

THEOREM 33. *Let* (d, f) *be a bipartite sequence, where* $d = (d_1, \ldots, d_p)$, $f = (f_1, \ldots, f_q)$, *and* $d_1 \geq d_2 \geq \cdots \geq d_p$, $p \geq f_1 \geq f_2 \geq \cdots \geq f_q$. *Then*

$$R(d, f) = \delta(f^*, d),$$

where $f^* = (f_1^*, f_2^*, \ldots, f_p^*)$.

Proof. By Theorem 30, we have $d \preccurlyeq f^*$. By Theorem 31, equality holds if and only if (d, f) is a difference sequence. Therefore $R(d, f)$ is the number of reverse unit transformations required to transform d into f^*. This is the same as $U(f^*, d)$, which equals $\delta(f^*, d)$ by Lemma 7. ∎

7. RELATED RESULTS

In this section we consider graphs whose degree sequences have a majorization gap of 1. Although we cannot characterize them, we show that they are both bithreshold and cobithreshold, as defined below.

Let $G = (V, E)$ be a graph, and let $i, j, k \in V$ such that $\deg(k) \geq \deg(i)+2$, $(i, j) \notin E$, $(k, j) \in E$. Then the operation of dropping from G the edge (k, j) and adding the edge (i, j) is called a **just rotation** from k to i [15] (see Figure 10).

Define an **unjust rotation** to be the reverse operation: Let $G = (V, E)$ be a graph, and let $i, j, k \in V$ such that $\deg(k) \geq \deg(i)$, $(i, j) \in E$, $(k, j) \notin E$. Then the operation of dropping from G the edge (i, j) and adding the edge (k, j) is an unjust rotation from i to k. Note that a just rotation performs a unit transformation on the degree sequence of the graph, and an unjust rotation performs a reverse unit transformation. The converse is also true:

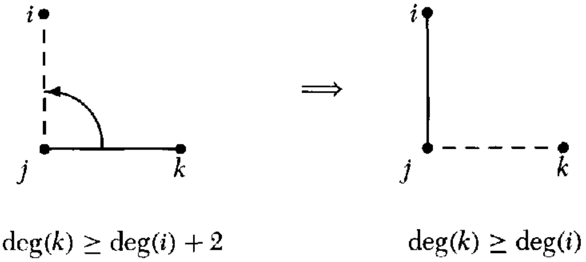

FIG. 10. Illustrating a just rotation from k to i. A solid line represents the presence of an edge, and a dashed line its absence.

LEMMA 34. [15] *Let d be the degree sequence of a graph G, and assume that d is obtained from another degree sequence e by a reverse unit transformation. Then there exist a graph H with degree sequence e and an unjust rotation that transforms H into G.*

LEMMA 35. (1) *If a graph G is obtained from a graph H by an unjust rotation, then $\overline{G}$ (the complementary graph of G) can also be obtained from $\overline{H}$ by an unjust rotation.*
(2) *Similarly for a just rotation.*

Proof. Follows from the definitions. ∎

The following theorem says that a degree sequence and its complementary degree sequence have the same majorization gap, as expected.

THEOREM 36. *Let $d = (d_1, \ldots, d_n)$ be a degree sequence with $d_1 \geq \cdots \geq d_n$, and let $\overline{d} = (\overline{d}_1, \ldots, \overline{d}_n)$ be the complementary degree sequence, where $\overline{d}_i = n - 1 - d_{n+1-i}$. Then*

$$R(d) = R(\overline{d}).$$

Proof. $C(\overline{d})$ is obtained from $C(d)$ by changing 1's into 0's and 0's into 1's, and reversing the order of the rows and columns. It follows that $\overline{d}_i' = n - 1 - d'_{n+1-i}$. It is then easy to prove that $\delta(d) = \delta(\overline{d})$. Hence $R(d) = R(\overline{d})$ by Theorem 5. ∎

It is also possible to prove Theorem 36 using Lemmas 34 and 35 and the fact that the complement of a threshold graph is a threshold graph. The details are omitted.

We define below (edge) unions and intersections only for graphs with the same vertex set.

A graph $G = (V, E)$ is called **bithreshold** if G is the intersection of two threshold graphs T_1 and T_2 such that every independent set of G is also independent in T_1 or T_2. The complement of a bithreshold graph is a **cobithreshold graph**: G is cobithreshold if G is the union of two threshold graphs T_1 and T_2 such that every clique of G is also a clique in T_1 or T_2. See [9] for more details.

A degree sequence d is called a **bithreshold** (a **cobithreshold**) sequence if there exists a bithreshold (a cobithreshold) graph with degree sequence d.

If $G = (V, E)$ is a graph and $w \notin V$ is a new vertex, we denote by $G + w$ the graph $(V \cup \{w\}, E)$ obtained by adding w as an isolated vertex, and by $G \circ w$ the graph $(V \cup \{w\}, E \cup \{wv : v \in V\})$ obtained by adding w as a dominating vertex. When $W \cap V = \emptyset$, we also extend the $+$ notation by $G + W = G + \sum_{w \in W} w$.

The following characterization of threshold graphs is well known.

THEOREM 37. [2] *A graph is a threshold graph if and only if it can be obtained from the one-vertex graph by the operations $+$ and $\circ$.*

THEOREM 38. *Let a graph H be obtained from a threshold graph G by a just rotation. Then*

(1) *H is cobithreshold;*
(2) *H is bithreshold.*

Proof. Let H be obtained from G by a just rotation from k to i. Let $N(p)$ denote the neighborhood of a vertex p in G, i.e.,

$$N(p) = \{q \in V : (p, q) \in E\}.$$

We also denote by G_S the subgraph of G induced by $S \subseteq V$. Define

$$\begin{aligned} T_1 &= G_{V-\{j\}} + j, \\ T_2 &= \left(G_{N(j)\cup\{i\}-\{k\}} \circ j\right) + k + \left[V - N(j) - \{i, j\}\right]. \end{aligned}$$

It is clear that T_1 and T_2 are threshold (by Theorem 37) and that $H = T_1 \cup T_2$. Let Q be a clique of H. If $j \notin Q$, then Q is a clique of T_1; otherwise Q is a clique of T_2. Hence H is cobithreshold. This fact, Theorem 36 and Lemma 35 imply that $\overline{H}$ is also cobithreshold, i.e., that H is bithreshold. ■

COROLLARY 39. *Let d be a degree sequence with $R(d) = 1$. Then d is bithreshold and cobithreshold.*

Proof. Let e be a threshold sequence obtained from d by a reverse unit transformation, and let H be the unique threshold graph with degree sequence e. By Lemma 34, some just rotation transforms H into a graph G with degree sequence d. G is bithreshold and cobithreshold by Theorem 38. ■

We thank the referee for useful comments.

REFERENCES

1 C. Berge, *Graphs and Hypergraphs*, North-Holland, Amsterdam, 1973.

2 V. Chvátal and P. L. Hammer, Aggregation of inequalities in integer programming, *Ann. Discrete Math.* 1:145–162 (1977).

3 P. Erdős and T. Gallai, Graphs with prescribed degrees of vertices (in Hungarian), *Mat. Lapok* 11:264–272 (1960).

4 L. R. Ford, Jr., and D. R. Fulkerson, *Flows in Networks*. Princeton U.P., Princeton, N.J., 1962.

5 D. Gale, A theorem on flows in networks, *Pacific J. Math.* 7:1073–1082 (1957).

6 M. C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Academic P., New York, 1980.

7 S. L. Hakimi, On the realizability of a set of integers as degrees of the vertices of a graph, *SIAM J. Appl. Math.* 10:496–506 (1960).

8 P. L. Hammer, T. Ibaraki, and B. Simeone, Threshold sequences, *SIAM J. Algebraic Discrete Methods* 2:39–49 (1981).

9 P. L. Hammer and N. V. R. Mahadev, Bithreshold graphs, *SIAM J. Algebraic Discrete Methods* 6:497–506 (1985).

10 P. L. Hammer, U. N. Peled, and X. Sun, Difference graphs, *Discrete Appl. Math.* 28:35–44 (1990).

11 G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge U.P., New York, 1952.

12 V. Havel, A remark on the existence of finite graphs (in Czech), *Časopis Pěst. Mat.* 80:477–480 (1955).

13 M. Koren, Extreme degree sequences of simple graphs, *J. Combin. Theory Ser. B* 15:213–224 (1973).

14 A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and its Applications*, Academic P., New York, 1979.

15 U. N. Peled and M. K. Srinivasan, The polytope of degree sequences, *Linear Algebra Appl.* 114:349–377 (1989).

16 H. J. Ryser, Combinatorial properties of matrices of zeros and ones, *Canad. J. Math.* 9:371–377 (1957).

17 H. J. Ryser, *Combinatorial Mathematics*, Carus Math. Monographs 14, Math. Assoc. Amer., distributed by Wiley, New York, 1963.

Received January 31, 1992; final manuscript accepted October 2, 1992